Solution to Problem Set 2 February 25, 2025

Exercise 1.

First, $S_n = \frac{1}{N-n} R_n$ is a function of $R_0, R_1, \ldots, R_n$. The triangle inequality and the fact that the random variables $X_1, \ldots, X_n$ are identically distributed give us

$$\begin{aligned}\mathbb{E}[|S_n|] &= \frac{1}{N-n}\mathbb{E}\left[\left|\sum_{i=1}^{N-n} X_i\right|\right] \\ &\leqslant \frac{1}{N-n}\sum_{i=1}^{N-n}\mathbb{E}[|X_1|] = \mathbb{E}[|X_1|] < +\infty,\end{aligned}$$

by hypothesis. Next, we need to show that $\mathbb{E}[S_{n+1}|R_0, \ldots, R_n] = S_n$ for all $n \in \mathbb{N}$. Using the definitions of the different terms, we have

$$\begin{aligned}\mathbb{E}[S_{n+1}|R_0, \ldots, R_n] &= \mathbb{E}[Y_{N-n-1}|Z_N, \ldots, Z_{N-n}] \\ &= \frac{1}{N-n-1}\mathbb{E}[Z_{N-n-1}|Z_N, \ldots, Z_{N-n}].\end{aligned}$$

Setting $m = N - n$, we obtain

$$\begin{aligned}&= \frac{1}{m-1}\mathbb{E}[Z_{m-1}|Z_m, \ldots, Z_N] \\ &= \frac{1}{m-1}\sum_{i=1}^{m-1}\mathbb{E}[X_i|Z_m, X_{m+1}, \ldots, X_N].\end{aligned}$$

We then observe that for all $i \leqslant m$,

$$\mathbb{E}[X_i|Z_m, X_{m+1}, \ldots, X_N] = \mathbb{E}[X_1|Z_m, X_{m+1}, \ldots, X_N]. \tag{1}$$

Indeed, knowing Z_m, the sum of m i.i.d. variables, the expectation of any individual variable in the sum is the same. Using this result, we obtain

$$\begin{aligned}\mathbb{E}[S_{n+1}|R_0, \ldots, R_n] &= \frac{1}{m-1}\sum_{i=1}^{m-1}\mathbb{E}[X_1|Z_m, X_{m+1}, \ldots, X_N] = \mathbb{E}[X_1|Z_m, X_{m+1}, \ldots, X_N] \\ &= \frac{1}{m}\sum_{i=1}^{m}\mathbb{E}\left[X_1|Z_m, X_{m+1}, \ldots, X_N\right] = \frac{1}{m}\mathbb{E}\left[\sum_{i=1}^{m} X_i|Z_m, X_{m+1}, \ldots, X_N\right] \\ &= \frac{1}{m}\mathbb{E}\left[Z_m|Z_m, X_{m+1}, \ldots, X_N\right] = \frac{1}{m}Z_m = R_{N-n} = S_n,\end{aligned}$$

which proves that (S_n) is a martingale with respect to (R_n).

It remains to show the equality (1). This follows directly from the Lemma in Exercise 5 of Problem set 1.

Exercise 2.

Firstly, note that $S_n = \frac{1}{n+2} R_n$ is a function of $R_0, R_1, \dots, R_n$. At any time n, there are $n+2$ balls in the urn. Thus, S_n is simply the proportion of red balls in the urn at time n. For all $n \in \mathbb{N}$, we have $S_n \leqslant 1$, so $\mathbb{E}[|S_n|] \leqslant 1$. Moreover,

$$\begin{aligned}
\mathbb{E}[S_{n+1}|R_0, \dots, R_n] &= \mathbb{E}\left[\frac{R_{n+1}}{n+3}|R_0, \dots, R_n\right] \\
&= \frac{1}{n+3}\mathbb{E}[R_{n+1}|R_0, \dots, R_n] \qquad (2)
\end{aligned}$$

It remains to compute $\mathbb{E}[R_{n+1}|R_0 = r_0, \dots, R_n = r_n]$ for all possible values of $r_0, \dots, r_n$. To do this, we use the definition of conditional expectation and the fact that, given R_n, the random variable R_{n+1} can take only two values: R_n and $R_n + 1$. Then,

$$\begin{aligned}
&\mathbb{E}[R_{n+1}|R_0 = r_0, \dots, R_n = r_n] \\
= &\sum_r r\,\mathbb{P}(R_{n+1} = r|R_0 = r_0, \dots, R_n = r_n) \\
= &r\,\mathbb{P}(R_{n+1} = r|R_0 = r_0, \dots, R_n = r_n) + (r+1)\mathbb{P}(R_{n+1} = r+1|R_0 = r_0, \dots, R_n = r_n) \\
= &r\,\frac{n+2-r}{n+2} + (r+1)\frac{r}{n+2} \\
= &\frac{n+3}{n+2}r.
\end{aligned}$$

Thus,

$$\mathbb{E}[R_{n+1}|R_0, \dots, R_n] = \frac{n+3}{n+2}R_n,$$

and, using this last relation and (2),

$$\mathbb{E}[S_{n+1}|R_0, \dots, R_n] = \frac{1}{n+3}\frac{n+3}{n+2}R_n = S_n.$$

Exercise 3.

Y_n is a function of X_n by the definition. Furthermore, since f is bounded, let $f_\infty \in \mathbb{R}^+$ be such that for all $x \in [0,1]$, $|f(x)| \leqslant f_\infty$. Thus,

$$\mathbb{E}[|Y_n|] \leqslant 2^n \mathbb{E}[|f(X_n + \frac{1}{2^n}) - f(X_n)|] \leqslant 2^n(f_\infty + f_\infty) < +\infty.$$

We start by noting that the conditional distribution of Z given $X_1, \dots, X_n$ is a uniform distribution on $[X_n, X_n + \frac{1}{2^n}[$. Indeed, given $x_1 \leqslant \cdots \leqslant x_n$ with $x_i = k_i 2^{-i}$ and $x_{i+1} \leqslant$

$x_i + 2^{-(i+1)}$, we observe that $\{X_1 = x_1, \dots, X_n = x_n\} = \{x_n \leqslant Z < x_n + 2^{-n}\}$. Let $I \subset [0,1]$ be an interval. Then

$$\mathbb{P}(Z \in I | X_1 = x_1, \dots, X_n = x_n) = \frac{\mathbb{P}(Z \in I, x_n \leqslant Z < x_n + 2^{-n})}{\mathbb{P}(x_n \leqslant Z < x_n + 2^{-n})}.$$

Consequently, if $I \cap [x_n, x_n + 2^{-n}[= \emptyset$, then this conditional probability is zero, and if $I \subset [x_n, x_n + 2^{-n}[$, it equals

$$\frac{\mathbb{P}(Z \in I)}{\mathbb{P}(x_n \leqslant Z < x_n + 2^{-n})} = \frac{|I|}{2^{-n}},$$

where $|I|$ is the length of I.

Thus, given $X_1, \dots, X_n$, the random variable X_{n+1} takes the value X_n or $X_n + \frac{1}{2^{n+1}}$ with conditional probability $\frac{1}{2}$ for each. Consequently,

$$\begin{aligned}
&\mathbb{E}[Y_{n+1} | X_1, \dots, X_n] = 2^{n+1}\mathbb{E}[f(X_{n+1} + \frac{1}{2^{n+1}}) - f(X_{n+1}) | X_1, \dots, X_n] \\
&= 2^{n+1}\left(\frac{1}{2}\left[f(X_n + \frac{1}{2^{n+1}}) - f(X_n)\right] + \frac{1}{2}\left[f(X_n + \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}}) - f(X_n + \frac{1}{2^{n+1}})\right]\right) \\
&= 2^n\left(f(X_n + \frac{1}{2^n}) - f(X_n)\right) \\
&= Y_n,
\end{aligned}$$

which proves that (Y_n) is a martingale relative to (X_n).

Exercise 4.

In what follows we prove the statements explicitly only for a supermartingale $(Y_n)_n$ and observe that the corresponding results for a submartingale follow from the fact that $(-Y_n)_n$ is a submartingale.

1. For any $n \in \mathbb{N}$ we have that $\mathbb{E}Y_{n+1} = \mathbb{E}Y_n$. Note that

   $$\mathbb{E}(Y_{n+1} - Y_n) = \mathbb{E}[\mathbb{E}[Y_{n+1} - Y_n | Z_1, Z_2, \dots, Z_n]] = 0 \tag{3}$$

   and that

   $$\mathbb{E}[Y_{n+1} - Y_n | Z_1, Z_2, \dots, Z_n] = \mathbb{E}[Y_{n+1} | Z_1, Z_2, \dots, Z_n] - Y_n \leqslant 0 \tag{4}$$

   from the fact that Y_n is a supermartingale, we have

   $$\mathbb{E}[Y_{n+1} | Z_1, Z_2, \dots, Z_n] - Y_n = 0, a.s..$$

   This implies that Y_n is a martingale.

2. Note that $x \mapsto \min(x, a)$ is a concave function. Hence, by conditional Jensen's inequality

   $$\mathbb{E}[\min(Y_{n+1}, a) | Z_1, \dots, Z_n] \leqslant \min(\mathbb{E}[Y_{n+1} | Z_1, \dots, Z_n], a) \leqslant \min(Y_n, a)$$

3. Let now $(Y_n)_n$ be equidistributed, then clearly $(\mathbb{E}[Y_n])_n$ is a constant sequence, and thus, by 1), $(Y_n)_n$ is a martingale. To show that $Y_1 = Y_2 = \dots$, we may use the fact that, same is true for, say, $(\min(Y_n, a))_n$. Observe that $A \subset B$ for $A, B \subset \Omega$ if and only if $A \cap B^c = \emptyset$. Therefore, our goal is to show that for any $p > n$,

$$\{Y_n \geqslant a\} \cap \{Y_p < a\} \quad \text{is a zero-set.}$$

By Exercise 1 and above observation that $(\min(Y_n, a) = Y_n \wedge a)_n$ is a martingale, we have

$$\mathbb{E}[(Y_p \wedge a)|_n] = Y_n \wedge a \quad \text{a.s.} \tag{5}$$

Here $|_n$ is a short version for $|Z_1, \dots, Z_n$. Furthermore,

$$\begin{aligned} a \cdot 1_{\{Y_n \geqslant a\}} = (Y_n \wedge a) 1_{\{Y_n \geqslant a\}} = \mathbb{E}[(Y_p \wedge a)|_n] 1_{\{Y_n \geqslant a\}} &= \mathbb{E}[(Y_p \wedge a) 1_{\{Y_n \geqslant a\}}|_n] \\ &= a \mathbb{E}[1_{\{Y_n \geqslant a, Y_p \geqslant a\}}|_n] + \mathbb{E}[Y_p 1_{\{Y_n \geqslant a > Y_p\}}|_n] \quad \text{a.s..} \end{aligned}$$

By taking expectation on both sides we get that

$$a\mathbb{P}[Y_n \geqslant a > Y_p] = a\mathbb{P}[Y_n \geqslant a] - a\mathbb{P}[Y_n \geqslant a, Y_p \geqslant a] = \mathbb{E}[Y_p \mathbb{1}_{\{Y_n \geqslant a > Y_p\}}],$$

which implies that $\mathbb{P}[Y_n \geqslant a > Y_p] = 0$. Otherwise the above is equivalent to

$$a = \mathbb{E}[Y_p | Y_n \geqslant a > Y_p] < a,$$

which leads to a contradiction.

In fact, this implies that $Y_p \geqslant Y_n$ almost surely. This is due to the fact that

$$\mathbb{P}[Y_n > Y_p] = \mathbb{P}[\exists a \in \mathbb{Q} : \; Y_n \geqslant a > Y_p] \leqslant \sum_{a \in \mathbb{Q}} \mathbb{P}[Y_n \geqslant a > Y_p] = 0.$$

By applying the argument to $(-Y_n)_n$, which is again an equidistributed martingale, we analogously obtain that $-Y_p \geqslant -Y_n$ almost surely. Combined together, $\mathbb{P}[Y_n = Y_p] = 1$ for all $n < p \in \mathbb{N}$. Since a countable union of zero sets is again a zero set we can conclude that

$$\mathbb{P}[Y_n = Y_1 \; \forall n \in \mathbb{N}] = 1.$$